УДК 616.711.002.07

ELEMENTS OF FUNCTIONAL ANALYSIS IN THE COURSE OF MATHEMATICS

G.M. Kulikov, A.D. Nakhman

Tambov State Technical University, Tambov

Represented by Doctor of Pedagogical Sciences, Professor N.P. Puchkov

Key words and phrases: a contraction operator; iterative process; linearly independent system of elements; linear normed space.

Abstract: The concept of block content “Elements of functional analysis” in the course of mathematics for students of engineering specialties (directions) has been introduced; applications to differential equations have been proposed.

Introduction

Modern mathematics and its applications are full of ideas and methods of functional analysis. The study of elements of functional analysis in the natural cycle of mathematical disciplines is prescribed by the state educational standards (SES) training for engineers. At the same time it is appropriate under the federal standards (GEF), bachelor's and master's of engineering and other areas. The student`s development of necessary concepts and facts contributes to the establishment and development of common cultural competence (the ability to synthesize, analyze, perception of information), and professional competence (the ability to analyze the state and dynamics of objects, systems using models of objects, and build the appropriate model of objects, the ability to properly formulate the objectives of its activities, to establish their relationship, etc.).

However, the process of the functional analysis studying in engineering universities is associated with the resolution of a number of existing contradictions, the most important of which seem to be for us the contradictions between:

– a significant amount of the relevant mathematical material and lack of training hours devoted to its study;

– lack of meaningful content previously studied by students of the courses of algebra and mathematical analysis, the provisions of which are summarized and developed in functional analysis.

Куликов Геннадий Михайлович – доктор физико-математических наук, профессор, заведующий кафедрой «Прикладная математика и механика»; Нахман Александр Давидович – кандидат физико-математических наук, доцент кафедры «Прикладная математика и механика», e-mail: alextmb@mail.ru, ТамбГТУ, г. Тамбов.

We can solve the problem of research, being undertaken in this article, with the resolution of these contradictions by optimizing the presentation way and amount of the required mathematical material.

The purpose of the content of the block, "Elements of functional analysis" is to familiarize students with basic concepts and facts, the simplest analysis in infinite-dimensional linear spaces and possibilities of its applications.

The process of learning is aimed at achieving the following objectives of general study:

– formation of a stable interest in the study of natural-mathematical sciences cycle, the development of creative abilities;

– developing skills of abstraction, the analysis and synthesis of objects;

– improvement in the use of mathematics.

As a result of studying the content of the block the student has to:

– get some basics of modern analysis in infinite-dimensional linear spaces and of the variety of specific implementations of common design;

– get the opportunity for further self-development and application of modern methods of analysis;

– expand mathematical view, raise the level of mathematical culture by working with objects of a higher level of abstraction (as compared to the analysis of finite-dimensional).

The content of the introductory course (including an illustration of the theoretical information with relevant examples) is possible if students understood the elements of linear algebra and analytical geometry, calculus (continuity, differentiability), complex numbers [1]. In the course of the study it is summarized the elements of the theory of linear operators in finite-dimensional spaces, the concept of limit of a sequence and function, and other concepts of finite-dimensional analysis. The proposed content of the following introductory courses are offered, in particular, theoretical exercises for independent solving by students to promote, in our opinion, a better absorption rate.

1. Linear space

We reject the traditional sequential presentation of concepts of metric and linear spaces (see, e.g. [2]), connecting them to the concept of a normed linear space.

1.1. Linear (vector) space is defined as a non-empty set $\mathcal{L}$ of elements $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ which have introduced the following two operations on its elements (vectors).

1) For any two elements $\mathbf{x}$, $\mathbf{y} \in \mathcal{L}$ is uniquely determined element $\mathbf{z} \in \mathcal{L}$ called their sum $\mathbf{x}+\mathbf{y}$, which obeys the axioms of commutativity and associativity of addition; at the same time:

– there exists an element $\mathbf{O} \in \mathcal{L}$ such that $\mathbf{x}+\mathbf{O}=\mathbf{x}$ for all $\mathbf{x} \in \mathcal{L}$ (the existence of zero);

– for each $\mathbf{x} \in \mathcal{L}$ there exists $-\mathbf{x} \in \mathcal{L}$ such that $\mathbf{x}+(-\mathbf{x})=\mathbf{O}$ (the existence of the opposite element).

2) For each $\mathbf{x} \in \mathcal{L}$ and every real number α the element of $\alpha\mathbf{x} \in \mathcal{L}$ is uniquely defined (the product of **x** by a number α). This operation is subject to the axioms of associativity (relative to the multiplication of numbers) and distributive (over addition of numbers), and under addition of elements), and to the axiom $1 \cdot \mathbf{x} = \mathbf{x}$.

The examples of linear spaces can be reproduced by:

a) the set of all vectors of the three-dimensional space V_3 that students are already familiar with; students are invited to independently verify that the usual operations of vector addition and multiplication by the number of vectors satisfying the above axioms;

b) the set $\mathbf{C}_{[a,b]}$ of all continuous functions on a given interval $[a, b]$.

Since the sum on the continuous functions *f* and *g* is a continuous function on $[a, b]$, the feasibility of the above axioms of addition and multiplication by a number derived from the properties of continuous functions on the interval (students are invited to check by themselves).

1.2. The concept of linear dependence (independence) of elements $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ of $\mathcal{L}$, in our opinion, is more easily comprehended by students if it is administered in terms of linear combinations. Thus, the element $x \in \Lambda$ is called a linear combination of elements, if there are real numbers $\alpha_1^*, \alpha_2^*, \ldots, \alpha_{k-1}^*$ such that

$$\mathbf{x}_k = \alpha_1^* \mathbf{x}_1 + \alpha_2^* \mathbf{x}_2 + \ldots + \alpha_{k-1}^* \mathbf{x}_{k-1}.$$

Now the elements $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ of $\mathcal{L}$ are called linearly independent if none of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ can be expressed as a linear combination of the others. An infinite sequence of elements $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, \ldots$ of $\mathcal{L}$ is linearly independent, if every finite subsequence is linearly independent.

Next, let us introduce the concept of finite (infinite)-dimensional spaces and the basis of:

a) if in the space $\mathcal{L}$ there is *n* linearly independent elements, but any element $n+1$ of this space is linearly dependent, then $\mathcal{L}$ is finite-dimensional and it is *n-dimensional*; if in $\mathcal{L}$ one can find a sequence of arbitrary finite number of linearly independent elements, the space is infinite-dimensional;

b) basis in a finite (*n*-dimensional) space $\mathcal{L}$ is any sequence of its elements $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ that are linearly independent; if

$$\mathbf{x} = \alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2 + \ldots + \alpha_n \mathbf{b}_n, \tag{1.1}$$

in this case the element **x** it is said to be decomposed in the basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$; while the assertion that the expansion (1.1) is always only be justified by the students themselves as a theoretical exercise.

The example of the space of all continuous functions $\mathbf{C}_{[a,b]}$ on the interval $[a, b]$ is really important for applications; the space is infinite since the sequence of power functions

$$1,\ x,\ x^2,\ \ldots,\ x^n$$

is linearly independent for any positive integer (arbitrarily large) n.

To introduce the concept of an expansion in basis in the infinite-dimensional spaces, it should be given a definition of convergence (to a certain element) series (infinite sum), composed of the elements of this space. To do this, in turn, it is required the concept of the norm of the element introduced below.

1.3. Non-empty set $\mathcal{L}'$, $\mathcal{L}' \subset \mathcal{L}$ is a subspace of $\mathcal{L}$, if it is itself a linear space with respect to certain linear operations in $\mathcal{L}$.

Thus, the subspace $\mathbf{C}_{[a,b]}$ with respect to a class $\mathbf{C}^{(n)}_{[a,b]}$ of all functions with continuous derivatives on $[a,b]$ up to the n-th order (inclusive).

2. The operators in linear spaces. Normed spaces

2.1. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two linear spaces. Compliance with A, assigning to $\mathbf{x} \in \mathcal{L}_2$ each single element $\mathbf{x} \in \mathcal{L}_2$ is called an operator acting from $\mathcal{L}_1$ within $\mathcal{L}_2$, So, the designation $\mathbf{y} = A\mathbf{x}$ is applied.

The operator A (acting from $\mathcal{L}_1$ within $\mathcal{L}_2$) is called linear if for any real numbers α_1, α_2, and any $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{L}_1$ it is held

$$\mathbf{A}(\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2) = \alpha_1\mathbf{A}\mathbf{x}_1 + \alpha_2\mathbf{A}\mathbf{x}_2. \qquad (2.1)$$

As an example, we can use a linear differential operator $\mathbf{L}$, acting out $\mathbf{C}^{(n)}_{[a,b]}$ in $\mathbf{C}_{[a,b]}$:

$$w = \mathbf{L}y, \quad \mathbf{L}y = y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y,$$

where

$$y \in \mathbf{C}^{(n)}_{[a,b]} \quad \text{and} \quad p_0, p_1, \ldots, p_{n-1} \in \mathbf{C}_{[a,b]}.$$

The property (2.1) follows from the linearity of differentiation

$$(\alpha_1 y_1(x) + \alpha_2 y_2(x))^k = \alpha_1 y_1^{(k)}(x) + \alpha_2 y_2^{(k)}(x), \qquad k = 1, 2, \ldots, n.$$

2.2. Let $\mathcal{L}$ be a linear space. The operator $\mathbf{A}\mathbf{x} = \|\mathbf{x}\|$, acting from $\mathcal{L}$ in R (the space of real numbers) is called a norm if the following conditions are fulfilled:

1) $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{O}$;
2) $\|\alpha\mathbf{x}\| = |\alpha| \cdot \|\mathbf{x}\|$ for every real number α and for any $\mathbf{x} \in \mathcal{L}$;
3) $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{L}_1$ (triangle inequality).

The linear space $\mathcal{L}$ in which is set a norm is called normalized, and the value $\|\mathbf{x}\|$ to associate an element $\mathbf{x}$ can be taken as a norm of this element.

In the space V_3 of all vectors of the three-dimensional space the norm of each vector $\{a_x, a_y, a_z\}$ can be regarded as its length

$$\|\boldsymbol{a}\| = |\boldsymbol{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}.$$

This property 1) is clear for the norm, the property 2) follows from the definition of multiplication of a vector by a number, and the inequality $\|\boldsymbol{a} + \mathbf{b}\| \le \|\boldsymbol{a}\| + \|\mathbf{b}\|$ here is a well known property of the triangle.

Since every continuous function $x = x(t)$ on the segment $[a, b]$ has the greatest value at a given interval, then the norm in the space $\mathbf{C}_{[a,b]}$ can be defined as

$$\|x\| = \max_{t\in[a,b]} |x(t)|.$$

Indeed, properties 1) and 2) for the norm are obviously fulfilled. Further, in view of the properties of the module we have

$$\|x_1 + x_2\| = \max_{t\in[a,b]} |x_1(t) + x_2(t)| \le \max_{t\in[a,b]} (|x_1(t)| + |x_2(t)|) \le$$
$$\le \max_{t\in[a,b]} |x_1(t)| + \max_{t\in[a,b]} |x_2(t)| = \|x_1\| + \|x_2\|.$$

for all $x_1, x_2 \in \mathbf{C}_{[a,b]}$.

2.3. The sequence $\{\mathbf{x}_n : \mathbf{x}_n \in \mathcal{L},\ n = 1, 2, \ldots\}$ is said to be convergent to the element $\mathbf{x} \in \mathcal{L}$ according to the norm of space $\mathcal{L}$ if

$$\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}\| = 0.$$

Particularly, we will use the vector sequence $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n, \ldots\}$ of infinite-dimensional linear normed space $\mathcal{L}$ as a basis, if any subsequence of these vectors is linearly independent; and we will expand the $\mathbf{x}$ in this basis and write as follows:

$$\mathbf{x} = \alpha_1\mathbf{b}_1 + \alpha_1\mathbf{b}_2 + \ldots \alpha_n\mathbf{b}_n + \ldots,$$

if

$$\lim_{n\to\infty} \|\mathbf{x} - (\alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \ldots \alpha_n\mathbf{b}_n)\| = 0$$

for some sequence of real numbers $\{\alpha_n,\ n = 1, 2, \ldots\}$.

2.4. Following problems may be proposed as theoretical exercises on the material of unit 2.

1) Prove that if each function $y_1, y_2, \ldots, y_\kappa$ satisfies the equation $Ly = 0$ (where L is linear differential operator of order *n*), then any linear combination satisfies this equation.

The above exercise can be solved by the students who do not possess yet the basic concepts and facts of differential equations [3].

2) Prove that the class of functions $\{x(t)\}$ on $[-a, a]$ with the bounded (by the same constant) derivatives of any order to form a linear space with the norm $\| x \| = \max_{t\in[-a,a]} | x(t) |$ and the basis

$$1, x, x^2, \dots, x^n, \dots$$

The solution of this exercise assumes that students possess the Maclaurin expansions of functions infinitely differentiable at the origin and its neighborhood.

3. A contraction mappings and its applications

We suggest introducing the well-known principle of contraction mappings (Banach principle, see, e.g. [2]) as follows.

3.1. The concept of a fixed point *x* of operator *A*, acting in a normed linear space Λ is introduced: *Ax*=*x*; then is considered to be a continuous operator at point *x*, such that

$$\lim_{n\to\infty} Ax_n = x, \quad \text{if} \quad x_n \to x$$

and the concept of completeness of the space Λ (every Cauchy sequence x_n has a limit $x \in \Lambda$) is also introduced.

3.2. Operator A is said to be a contraction if there exists a number $q \in (0,1)$ such that $\| Ax - Ay \| \le q \| x - y \|$.

3.3. It is proved:

Theorem 1. Contraction operator *A* is continuous at each point $x \in \Lambda$.

Theorem 2. If the contraction operator *A* maps a complete space Λ then it has a unique fixed point.

The main proof steps of Theorem 2 are:

a) the construction of the sequence $\{x_n\}$ in the form

$$x_1 = Ax_0,\ x_2 = Ax_1,\ \dots,\ x_n = Ax_{n-1}, \dots,$$

Where $x_0 \in \Lambda$ is an arbitrary point;

b) the proof of its fundamental nature, which is based on obtaining the estimate

$$\| x_m - x_n \| \le \frac{q^n}{1-q} \| x_1 - x_0 \|;$$

c) the use of the continuity of *A* is for a fixed point receipt, and, finally, proof of its uniqueness.

As an application of the principle of contraction mappings, the iterative process is considered

$$y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n(x))dx$$

for solutions of the Cauchy problem

$$\begin{cases} y' = f(x, y); \\ y(x_0) = y_0. \end{cases}$$

for the first order differential equation.

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Элементы функционального анализа в курсе математики

Г.М. Куликов, А.Д. Нахман

ФГБОУ ВПО «Тамбовский государственный технический университет», г. Тамбов

Ключевые слова и фразы: итерационный процесс; линейно независимые системы элементов; линейное нормированное пространство; сжимающий оператор.

Аннотация: Излагается концепция блока содержания «Элементы функционального анализа» в курсе математики для студентов инженерных специальностей (направлений), и предлагаются его приложения в дифференциальных уравнениях.
